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THE DISCRIMINATION THEOREM FOR NORMALITY TO NON-INTEGER BASES

BY

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ABSTRACT

In a previous paper, [7], the authors together with Gavin Brown gave a complete description of the values of θ, r and s for which numbers normal in base θ^r are normal in base θ^s. Here θ is some real number greater than 1 and x is normal in base θ if $\{\theta^n x\}$ is uniformly distributed modulo 1. The aim of this paper is to complete this circle of ideas by describing those ϕ and ψ for which normality in base ϕ implies normality in base ψ. We show, in fact, that this can only happen if both are integer powers of some base θ and are thus subject to the constraints imposed by the results of [7]. This paper then completes the answer to the problem raised by Mendès France in [12] of determining those ϕ and ψ for which normality in one implies normality in the other.

Introduction

Let θ be a real number greater than 1 (a **base**). We write $B(\theta)$ to indicate the set of those real numbers x which are normal in base θ; that is, $(\{x\theta^n\})$ is uniformly distributed modulo 1, where $\{x\}$ denotes the fractional part of x. We are concerned with the question of when $B(\theta)$ is contained in $B(\phi)$ for bases θ

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and ϕ. This problem was first raised for integer bases (in fact, for the bases 2 and 3) by Steinhaus [19] in the New Scottish Book. The specific problem of Steinhaus was solved by Cassels [8] and Schmidt [17] independently. Schmidt showed that, for any pair of integer bases r and s, either there are positive integers p and q such that $r^p = s^q$ or there are uncountably many real numbers normal in base r and non-normal in base s.

For non-integer bases the problem is made harder and different by the fact that $(\{x\theta^n\})$ are not the results of applying the map $x \mapsto \{x\theta\}$ iteratively to x. In [12], Mendès France asks whether $B(\phi) \subset B(\psi)$ implies that $\log\phi/\log\psi$ is rational and conjectures that this is indeed the case when ϕ and ψ are Pisot numbers. The aim of this paper is to prove the surprising result that this implication holds for *all* real numbers ϕ and ϕ greater than 1. Thus our main theorem is:

THEOREM 1: *Let ϕ and ψ be real numbers greater than* 1. *If $B(\phi) \subset B(\psi)$ then $\log\phi/\log\psi$ is rational.*

It follows that there is some θ such that $\phi = \theta^r$ and $\psi = \theta^s$, for positive integers r and s, so that this theorem, taken in conjunction with the main theorem of the paper of Brown, Moran and Pollington [7], yields the following complete description of the ϕ and ψ for which such an inclusion holds.

THEOREM 2 (cf. [7, Theorem 1]): *Let ϕ and ψ be real numbers greater than* 1. *Then an inclusion $B(\phi) \subset B(\psi)$ holds when and only when there exist θ, r and s such that $\phi = \theta^s$ and $\psi = \theta^r$ and either*

(1) *there exists k such that $\theta^k \in \mathbf{Z}$ and $\mathbf{Q}(\theta^r) \subset \mathbf{Q}(\theta^s)$; or*

(2) *there exists k such that $\theta^k \pm \theta^{-k} \in \mathbf{Z}$ and s divides r.*

The fact that $B(\theta) \subset B(\theta^r)$, for any positive integer r, when θ is such that $\theta^k \pm \theta^{-k}$ is in $\mathbf{Z}$ is due to Bertrand ([1] and [2]).

The proof of Theorem 1 uses techniques similar to those found in [4]. Like all previous results of this nature (cf. [3], [4], [5], [6], [7], [10], [13], [14], [15], [17], [18]) the key is to find a probability measure μ (a **discriminatory measure**) which assigns zero mass to $B(\psi)$ and full mass to $B(\phi)$, except, of course, when the conditions on ϕ and ψ are met. The innovation introduced by Brown, Moran and Pearce in [4] was to use Riesz product measures in place of the Cantor-type measures used by other authors. Here again we shall use Riesz products as well as refinements of arguments presented in [7]. The proof, however, introduces new techniques not found in previous papers, mainly to do with the choice of the terms in the Riesz product.

The proof given here can be readily adapted (in fact with some simplifications) to give a new proof of the original theorem of Schmidt for integer bases. This proof, unlike the original Riesz product proof, avoids invoking Baker's Theorem on linear forms in logarithms, by explicitly using the continued fraction expansion of $\log\psi/\log\phi$.

Preliminaries

We begin by describing Riesz product measures. Rather than striving for full generality, we restrict ourselves to such measures of the specific kind that we need to prove the theorem. A more complete description of these measures and their properties may be found in [9].

First choose p such that $\psi^p > 3$ and a set A of positive integers whose upper density is positive, that is,

$$\limsup_{N\to\infty} \frac{1}{N}\{n \in A\colon n \leq N\} > 0. \tag{1}$$

This set will be chosen later to fit the specific problem. Now consider the function

$$F_N(t) = \prod_{n\in A, n\leq N} \left(1+\cos(2\pi\psi^{pn}t)\right)\frac{(1-\cos t)}{\pi t^2}.$$

This is positive and has integral $\int_{-\infty}^{\infty} F_N(t)\,dt = 1$. Thus the measure μ_N whose Radon-Nikodym derivative with respect to Lebesgue measure on $\mathbf{R}$ is F_N is a probability measure. It can be shown, by considering Fourier transforms and the decay of the measure, that the sequence (μ_N) converges in the weak$*$ topology to a measure μ. This is the Riesz product measure which we shall use. Its salient feature is that its Fourier transform satisfies

$$\hat{\mu}(\gamma) = 0$$

unless

$$\left|\gamma - \sum_{i\in A} \varepsilon_i\psi^{pi}\right| < 1,$$

where $\varepsilon_i = 0, \pm 1$. For such ε_i,

$$\hat{\mu}\left(\sum_{i\in A}\varepsilon_i\psi^{pi}\right) = \left(\frac{1}{2}\right)^{\sum_i |\varepsilon_i|} \tag{2}$$

and the remainder of the Fourier transform is obtained by linear interpolation. As we have indicated earlier, such measures are defined and studied in [8].

We need to show that this is, indeed, a discriminatory measure; that is, that $\mu(B(\psi)) = 0$ and $\mu(B(\phi)) = 1$. The former is relatively straightforward.

LEMMA 1: *For the measure μ defined above,*

$$\mu(B(\psi)) = 0.$$

Proof: Write $X_n(t) = e^{2\pi i \psi^{p^n} t}$. Then, by (2), these random variables are uncorrelated, that is $E(X_n \overline{X}_m) = E(X_n)E(\overline{X}_m)$, for all $n \neq m$. By the Strong Law of Large Numbers for such random variables,

$$\frac{1}{N}\sum_{n=1}^{N}(X_n - E(X_n)) \to 0$$

almost surely. However, by (1),

$$\limsup_{N\to\infty} \frac{1}{N}\left|\sum_{n=1}^{N} E(X_n)\right| = \frac{1}{2}\limsup_{N\to\infty}\frac{1}{N}\#\{n \in A : n \leq N\} > 0,$$

so that, for μ-almost all t,

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i \psi^{p^n} t} \not\to 0.$$

By Weyl's Criterion, t is not normal in base ψ almost everywhere with respect to μ. ∎

The problem of proving that, unless $\log\psi/\log\phi$ is rational, $\mu(B(\phi)) = 1$ will concern us throughout most of the remainder of this paper. To handle this, we shall need to use the following Lemma of Davenport, Erdős and LeVeque (see, for example, [16]). The particular form of this lemma we quote is tailored to the application we have in mind.

LEMMA 2: *Let (x_n) be a sequence of real numbers tending to infinity and let μ be a probability measure on $\mathbf{R}$ such that*

$$\sum_{N=1}^{\infty}\frac{1}{N^3}\sum_{n,m=1}^{N}\hat{\mu}(k(x_n - x_m)) < \infty \tag{3}$$

for all integers $k \neq 0$. Then $(x_n t)$ is uniformly distributed modulo 1 for μ-almost all t.

Proof of Theorem 1

The key idea is that in order that (3) should fail there will be many values of n and m such that an inequality of the form

$$\left| k(\phi^n - \phi^m) - \sum_{i \in A, i \leq R(n,m)} \varepsilon_i \psi^{pi} \right| < 1 \tag{4}$$

holds, and, in view of (2), in many of these inequalities most of the ε_i's will be zero. This will allow us to obtain too many good rational approximations for $\log \psi / \log \phi$. We assume throughout that $\varepsilon_{R(n,m)} \neq 0$

We need to quantify these statements. Before we can do that, it will be necessary to specify A. We shall assume that $\alpha = \log \psi / \log \phi$ is irrational to obtain a contradiction and that its continued fraction expansion is

$$\alpha = [a_0; a_1, a_2, \dots]$$

where, of course, all of the a_i's are non-zero. We let

$$\frac{p_r}{q_r} = [a_0; a_1, a_2, \dots, a_r]$$

denote the rth partial quotient. Now we define

$$A = \bigcup_r (q_r, 2q_r) \cap (q_r, q_{r+1}),$$

where (a, b) denotes the interval of integers d such that $a < d < b$. Observe that the sequence (q_r) increases at least exponentially, and that the upper density of A is positive, in fact, at least $1/4$.

Now we construct the Riesz product μ as indicated previously and apply the preceding lemmas. First we note that $\mu(B(\psi)) = 0$ by Lemma 1. Now we turn to the problem of showing that $\mu(B(\phi)) = 1$. If not then the sum in (3) is infinite and there are infinitely many pairs (m, n) for which (4) holds. In fact, it is convenient to write

$$\hat{\mu}\left(\sum_{j \in A} \varepsilon_j \psi^{pj} \right) = 2^{-r(n,m)}$$

where $r(n, m)$ is the number of non-zero ε_j's and

$$\left| k(\phi^n - \phi^m) - \sum_{i \in A, i \leq R(n,m)} \varepsilon_i \psi^{pi} \right| < 1. \tag{5}$$

We write $r(n,m)=\infty$ if no inequality of the form (5) exists. Note that

$$|\hat{\mu}(k(\phi^n-\phi^m))|\le \frac{1}{2^{r(n,m)}}.$$

Set $t(n)=n/(\log n)^2$. Then, under the assumption that the sum in (3) is infinite,

$$\sum_N \frac{1}{N^3}\sum_{0\le n\le N}\ \sum_{0\le m<n-t(n)}\frac{1}{2^{r(n,m)}}=\infty. \tag{6}$$

It follows that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{0\le m<n-t(n)}\frac{1}{2^{r(n,m)}}=\infty$$

and hence that

$$\sum_{2^S\le n<2^{S+1}}\ \sum_{0\le m<n-t(n)}\frac{1}{2^{r(n,m)}}\ge\frac{2^{2S}}{S^2}$$

for infinitely many S, say $S\in\Lambda$. For these S, then,

$$\sum_{0\le m<n-t(n)}\frac{1}{2^{r(n,m)}}\ge C\left(\frac{2^S}{S^3}\right),$$

for some constant $C>0$ and at least $2^S/S^3$ n's between 2^S and 2^{S+1}. Call the set of such n's $G(S)$. For each of these n's there are at least $C'2^S/S^4$ m's satisfying

$$0\le m\le n-t(n)\quad\text{and}\quad r(n,m)\le 2\log S,$$

for some constant $C'>0$. We nominate one such $m=m(n)$ for each n in $G(S)$. Fix $n\in G(S)$ and suppose that (5) holds for $m=m(n)$. Then, somewhere in the sequence

$$\varepsilon_{R(n)},\varepsilon_{R(n)-1},\dots,\varepsilon_{R(n)-v(S)},$$

where $R(n)=R(n,m(n))$ and $v(S)=2S^2\log S$, there is a block of S^2 zeros.

Let us write $V(n)$ for the largest i such that ε_i is followed by a block of at least S^2 zeros. Since $R(n)-V(n)$ is less than $v(S)$ and $r(n,m(n))\le 2\log S$, the number of possible choices of

$$\varepsilon_{R(n)},\varepsilon_{R(n)-1},\dots,\varepsilon_{V(n)}$$

does not exceed

$$r(S)=(2v(S))^{2\log S}\le\frac{2^{S/2}}{S^3}$$

for large enough S. It follows that, for sufficiently large S and for some

$$w = \sum_{k=0}^{M} \varepsilon'_k \psi^{-pk},$$

where $\varepsilon'_k = 0, \pm 1$, there are at least $2^{S/2}$ n's between 2^S and 2^{S+1} for which

$$k\phi^n(1+\tau_n) = w\psi^{pR(n)}(1+\sigma_n) \tag{7}$$

where

$$|\tau_n| \le \phi^{-t(n)} \le e^{-Cn(\log n)^{-2}},$$
$$|\sigma_n| \le e^{-C'(\log n)^2},$$

for some positive constants C and C', and $R(n) \in A$. Taking logarithms in (7), we obtain

$$n\log\phi + \log k - \log w = pR(n)\log\psi + \omega_n, \tag{8}$$

where $|\omega_n| = O(e^{-C(\log n)^2})$. We now use one such value n_0, the smallest, to eliminate the constant term from (8) and obtain, for $2^{S/2}$ n's between 2^S and 2^{S+1}, that

$$(n - n_0)\log\phi = p(R(n) - R(n_0))\log\psi + \omega_n - \omega_{n_0}. \tag{9}$$

For these n's, setting $v_n = n - n_0$ and $u_n = p(R(n) - R(n_0))$, we have

$$\frac{\log\phi}{\log\psi} = \frac{u_n}{v_n} + \delta_n$$

where

$$v_n \le 2^S \quad \text{and} \quad |\delta_n| \le \exp(-CS^2).$$

Since, by (8), $R(n) \le C'n$ for some positive constant C', it follows that

$$|\delta_n| \le e^{-C(\log R(n))^2} < \frac{1}{R(n)^3} < \frac{1}{2v_n^2} \tag{10}$$

for sufficiently large S. By Lagrange's Theorem u_n/v_n must be a convergent of α. Consequently, there is some integer d so that $u_n = dp_r$ and $v_n = dq_r$ where

$$\frac{p_r}{q_r} = [a_0; a_1, \dots, a_r].$$

Moreover,

$$\frac{1}{R(n)^3} > \left|\alpha - \frac{p_r}{q_r}\right| > \frac{1}{q_r(q_r + q_{r+1})} > \frac{1}{2R(n)q_{r+1}}$$

(see [11, Theorem 9.9]), so that $R(n) < q_{r+1}$. Since $R(n) \in A$, it is less than $2q_r$ and $u_n < 2pq_r$. It follows that $v_n \leq Kq_r$ for some constant K independent of n. Now we have shown that there are at most K times as many n's in $G(S)$ as there are denominators q_r from the convergents of α in the interval of integers $[1, 2^S]$. Since the q_r's increase exponentially, this is at most CS for some constant C, and so contradicts the statement that $G(S)$ has $2^{S/2}$ elements. The assumption that α is irrational is false and the proof of the theorem is complete. ∎

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